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On the Topology of Lagrangian Milnor Fibres

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1 Introduction

Singularities of Lagrangian varieties are a fundamental object of study. They appear in various contexts (see [1, 6]). For example, they play a role in the theory of complete integrable systems and their quantisations (see [2, 4, 7]). In [10], a complex associated to a Lagrangian variety $L \subset \mathbb{C}^{2n}$, analogous to the Koszul-Chevalley complex in the theory of Lie algebras, was introduced. The first cohomology vector space of this complex computes the infinitesimal deformation of the Lagrangian variety, like for the case of Lie algebras.

In [4], a relative version of this complex was introduced for singular Lagrangian fibrations. Under some transversality conditions, it was proved in that paper that the higher direct image sheaves are coherent. In this paper, we prove that they are actually sheaves of *free* modules, provided that the deformation is Lagrangian infinitesimally versal over a smooth base. This result enables us to prove that for such deformations, the dimension of the first cohomology group of a Lagrangian Milnor fibre equals the codimension of the singularity, that is, the dimension of the base of a Lagrangian miniversal deformation.

2 Deformations of Lagrangian varieties

We recall briefly the construction of the complex of infinitesimal deformations of Lagrangian varieties, details can be found in [4, 5, 8, 9, 10].

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2.1 Lagrangian mappings

We consider the space $\mathbb{C}^{2n} = \{(q_1, \dots, q_n, p_1, \dots, p_n)\}$ endowed with the standard symplectic structure $\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$. We denote by X a domain in $\mathbb{C}^{2n}$. The Poisson bracket $\{f, g\}$ of two holomorphic functions $f, g : X \to \mathbb{C}$ can be defined by the formula

$$\{f, g\}\omega^n = df \wedge dg \wedge \omega^{n-1}. \tag{2.1}$$

Recall that a *Lagrangian submanifold* of $\mathbb{C}^{2n}$ is an n-dimensional holomorphic manifold on which the symplectic form vanishes. A Lagrangian variety is a purely n-dimensional analytic variety L such that the smooth part of L is a Lagrangian manifold.

Definition 2.1. A *Lagrangian mapping* is a holomorphic map $f = (f_1, \dots, f_n) : X \to \mathbb{C}^n$ such that

(1) the variety $\{f = 0\}$ is of pure dimension n,
(2) for all pairs i, j, $\{f_i, f_j\}$ belongs to the ideal generated by $f_1, \dots, f_n$.

The zero-fibre of a Lagrangian mapping is a Lagrangian variety $L \subset \mathbb{C}^{2n}$. On L, we define a stratification as follows. Denote by $X_1, \dots, X_n$ the Hamilton vector fields of $f_1, \dots, f_n$. Let $l(q, p)$ be the dimension of the vector space generated by the X_i's at p.

The stratum $L_j \subset L$ is defined by

$$L_j = \big\{(q, p) : l(q, p) = j\big\}. \tag{2.2}$$

We have that $L = \bigcup_{j=0}^{n} L_j$.

Definition 2.2. The Lagrangian mapping f is called *nondegenerate* if, for any k, the variety L_k is of dimension at most k.

Remark 2.3. For $n = 1$, the nondegeneracy condition means that the origin is an isolated singular point of the plane curve germ $\{f = 0\}$. This notion was introduced in [10, condition (P)], it can be considered as the symplectic analog of the notion of isolated singularity of a complex hypersurface.

There exists an obvious notion of a *Lagrangian deformation* of a Lagrangian mapping $f : X \to \mathbb{C}^n$ with parameter space Λ: it is a holomorphic map

$$F = \big(F_1, \dots, F_n\big) : \Lambda \times X \longrightarrow \mathbb{C}^n \tag{2.3}$$

with $F(0, \cdot) = f(\cdot)$ and such that the Poisson brackets $\{F_i, F_j\}$ with respect to the (q, p) variables belong to the ideal generated by the F_k's.

There are obvious notions of equivalence and versality of deformations that we will not spell out here.

2.2 The complex of Lagrangian infinitesimal deformations

Let $f = (f_1, \dots, f_n) : (\mathbb{C}^{2n}, 0) \to (\mathbb{C}^n, 0)$ be a Lagrangian mapping germ. Denote by $\mathcal{O}_{2n}$ the ring of germs at $0 \in \mathbb{C}^{2n}$ of holomorphic functions and by $\mathcal{O}_{L,0}$ the quotient ring of $\mathcal{O}_{2n}$ by the ideal I generated by f_j's. As for a complete intersection, one has $I/I^2 \approx \mathcal{O}_{L,0}^n$, the complex of Lagrangian infinitesimal deformations [10] which, in general, has terms $C_f^{\cdot} = \mathrm{Hom}(\bigwedge^{\cdot} I/I^2, \mathcal{O}_{L,0})$ takes the form

$$C_f^{\cdot} : 0 \longrightarrow \mathcal{O}_{L,0} \longrightarrow \bigwedge^1 \mathcal{O}_{L,0}^n \longrightarrow \bigwedge^2 \mathcal{O}_{L,0}^n \longrightarrow \cdots \longrightarrow \bigwedge^n \mathcal{O}_{L,0}^n \longrightarrow 0. \tag{2.4}$$

We consider the particular case $n = 2$ and refer to [8, 10] for the general case.

We use the identifications

$$\begin{aligned} &\bigwedge^1 \mathcal{O}_{L,0}^2 \approx \mathcal{O}_{L,0}^2, \\ &\bigwedge^2 \mathcal{O}_{L,0}^2 \approx \mathcal{O}_{L,0}. \end{aligned} \tag{2.5}$$

Since f is a Lagrangian complete intersection, there exist $a, b \in \mathcal{O}_{2n}$ such that

$$\{f_1, f_2\} = af_1 + bf_2. \tag{2.6}$$

We define the first differential to be

$$\begin{aligned} &\delta : \mathcal{O}_{L,0} \longrightarrow \mathcal{O}_{L,0}^2, \\ &h \longmapsto (\{h, f_1\}, \{h, f_2\}) \end{aligned} \tag{2.7}$$

and the second differential by

$$\begin{aligned} &\delta : \mathcal{O}_{L,0}^2 \longrightarrow \mathcal{O}_{L,0}, \\ &(m_1, m_2) \longmapsto \{m_1, f_2\} + \{f_1, m_2\} - am_1 - bm_2. \end{aligned} \tag{2.8}$$

(In the definition of the differential, we abusively denoted a function in $\mathcal{O}_{2n}$ and its projection in the factor ring $\mathcal{O}_{L,0}$ by the same symbol.)

It is readily verified that the first cohomology space of the complex $C_f^{\cdot}$ is equal to the first-order Lagrangian deformations of f modulo infinitesimally trivial deformations, where the coordinate changes have to be symplectic.

The following result is due to Sevenheck and van Straten [10] (it is in fact valid for more general Lagrangian varieties).

Theorem 2.4. If f is a nondegenerate Lagrangian mapping germ, then the vector spaces $H^k(C^{\cdot}_f)$ are finite dimensional. □

2.3 The relative complex of Lagrangian deformations

There is no difficulty in sheafifying the construction of the complex of Lagrangian deformations to obtain a complex of sheaves. There is also no difficulty in introducing a relative complex of Lagrangian deformations, that is, a complex with parameters.

Let $\Phi : (\mathbb{C}^k \times \mathbb{C}^{2n}, 0) \to (\mathbb{C}^n, 0)$ be a deformation of a Lagrangian mapping germ f. In [4], it is explained how to construct the so-called *standard representatives* Λ for $\mathbb{C}^k$, X for $\mathbb{C}^{2n}$, B for $\mathbb{C}^n$, and $F : \Lambda \times X \to B$ for Φ that are appropriate for this situation. The relative complex of Lagrangian deformations associated to F is supported on the subvariety $Y \subset (\Lambda \times X)$ defined by

$$Y = \{(\lambda, x) \in \Lambda \times X : F(\lambda, x) = 0\}, \qquad F = (F_1, \dots, F_n). \tag{2.9}$$

Therefore, we will denote the complex of Lagrangian deformation associated to F by $\mathcal{C}^{\cdot}_{Y/\Lambda}$.

The map

$$\begin{aligned} \varphi : Y &\longrightarrow \Lambda, \\ (\lambda, x) &\longrightarrow \lambda \end{aligned} \tag{2.10}$$

will, abusively, be called a *standard representative* of the deformation Φ.

Finally, remark that the symplectic structure on X defines a Poisson structure on $\Lambda \times X$ and that the Hamilton vector fields of the components $F_1, \dots, F_n$ of F are tangent to the fibres of φ.

In [4], the following coherence theorem was proved.

Theorem 2.5. If φ is a standard representative of a deformation of a nondegenerate Lagrangian mapping germ, then the following properties hold:

(a) the sheaves $\mathbb{R}^p \varphi_* \mathcal{C}^{\cdot}_{Y/\Lambda}$ are coherent sheaves of $\mathcal{O}_\Lambda$-modules;

(b) there is a canonical isomorphism of $\mathcal{O}_{\Lambda,0}$-modules

$$\left(\mathbb{R}^p \varphi_* \mathcal{C}^{\cdot}_{Y/\Lambda}\right)_0 \approx H^p\left(\mathcal{C}^{\cdot}_{Y/\Lambda,0}\right). \tag{2.11}$$

□

2.4 The Lagrangian $\tau = \mu$ theorem

Before formulating the theorem, we need some definitions.

Definition 2.6. A deformation $\Phi : (\mathbb{C}^k \times \mathbb{C}^{2n}, 0) \to (\mathbb{C}^n, 0)$ of a Lagrangian complete intersection germ $f : (\mathbb{C}^{2n}, 0) \to (\mathbb{C}^n, 0)$ is called *infinitesimally $\mathcal{L}$-versal* if the cohomology classes of the restrictions of the $\partial_{\lambda_j}\Phi$'s to $\lambda = 0$ generate $H^1(C_f^{\cdot})$.

A *Milnor fibre* L_λ of a deformation $\Phi : (\mathbb{C}^k \times \mathbb{C}^{2n}, 0) \to (\mathbb{C}^n, 0)$ of a Lagrangian mapping germ f is a fibre $L_\lambda = \{\varphi^{-1}(\lambda)\}$ of a standard representative $\varphi : Y \to \Lambda$ of Φ which is smooth. If such smooth fibres occur, the deformation Φ is called a *smoothing* of $L = L_0$.

We denote by $\beta_1, \ldots, \beta_n$ the Betti numbers of a Milnor fibre of the deformation Φ.

Theorem 2.7. If the deformation Φ is an infinitesimally $\mathcal{L}$-versal deformation of f, then for any standard representative φ of Φ, the sheaf $\mathbb{R}^1\varphi_* \mathcal{C}^{\cdot}_{Y/\Lambda}$ is a sheaf of free $\mathcal{O}_\Lambda$-modules. If Φ is a smoothing, then the rank of this module is equal to β_1. □

This result implies the following Lagrangian version of the classical $\tau = \mu$ theorem for isolated hypersurface singularities.

Corollary 2.8. The vector space $H^1(C_f^{\cdot})$ is of dimension β_1 provided that there exists a smoothing Φ of L. □

Example 2.9. Consider the Lagrangian mapping germ

$$f = (f_1, \ldots, f_n) : (\mathbb{C}^{2n}, 0) \longrightarrow (\mathbb{C}^n, 0) \tag{2.12}$$

defined by

$$f_i = p_i q_i. \tag{2.13}$$

A straightforward computation shows that the deformation

$$\Phi = (\Phi_1, \ldots, \Phi_n) : (\mathbb{C}^n \times \mathbb{C}^{2n}, 0) \longrightarrow (\mathbb{C}^n, 0) \tag{2.14}$$

defined by

$$\Phi_i = p_i q_i + \lambda_i \tag{2.15}$$

is infinitesimally $\mathcal{L}$-versal. Thus, as can be seen directly, the Milnor number of a Lagrangian fibre is equal to n.

One may conjecture that more generally the sheaves $\mathcal{H}^k(\mathcal{C}^{\cdot}_{Y/\Lambda})$ are sheaves of free $\mathcal{O}_\Lambda$-modules of rank β_k, which will imply that the vector space $H^k(C^{\cdot}_f)$ is of dimension β_k for germs of smoothable Lagrangian varieties.

3 Proof of Theorem 2.7

3.1 The sheaf $R^1\varphi_*\mathcal{C}_{Y/\Lambda}$ is a sheaf of free modules

With the same notations as before, let $\lambda_1, \lambda_2, \dots, \lambda_k$ be a system of coordinates on Λ near 0 and put $\Lambda_0 = \Lambda$ and

$$\Lambda_p = \left\{\lambda_1 = \lambda_2 = \cdots = \lambda_p = 0\right\} \subset \Lambda. \tag{3.1}$$

Let $\varphi^0 = \varphi$ and denote by φ^1 the restriction of φ above $\{\lambda_1 = 0\}$, that is,

$$\varphi^1 : Y_1 \longrightarrow \Lambda_1 \tag{3.2}$$

is defined as the projection of

$$Y_1 = \left\{\left(0, \lambda_2, \dots, \lambda_k, x\right) : F(\lambda, x) = 0\right\} \tag{3.3}$$

to Λ_1. In this way, we get a sequence of deformations of the Lagrange variety L: $\varphi^0, \dots, \varphi^k$, where φ^j is the restriction of φ above Λ_j. The sequence ends at φ^k which is the constant deformation

$$\varphi^k : L \longrightarrow \{0\} \tag{3.4}$$

of the Lagrangian variety L.

Denote by i_p the injective map

$$\begin{aligned} i_p : \mathcal{C}^{\cdot}_{Y_p/\Lambda_p} &\longrightarrow \mathcal{C}^{\cdot}_{Y_p/\Lambda_p}, \\ [m] &\longrightarrow \lambda_{p+1}[m] \end{aligned} \tag{3.5}$$

and by r_p the restriction map

$$\begin{aligned} r_p : \mathcal{C}^{\cdot}_{Y_p/\Lambda_p} &\longrightarrow \mathcal{C}^{\cdot}_{Y_{p+1}/\Lambda_{p+1}}, \\ [m] &\longrightarrow [m]_{|\lambda_{p+1}=0}. \end{aligned} \tag{3.6}$$

We assert that there is an exact sequence

$$0 \longrightarrow \mathcal{H}^1(\mathcal{C}^{\cdot}_{Y_p/\Lambda_p}) \xrightarrow{i_p^*} \mathcal{H}^1(\mathcal{C}^{\cdot}_{Y_p/\Lambda_p}) \xrightarrow{r_p^*} \mathcal{H}^1(\mathcal{C}^{\cdot}_{Y_{p+1}/\Lambda_{p+1}}) \longrightarrow 0. \tag{3.7}$$

Here $\mathcal{H}^i(-)$ denotes the cohomology sheaf of a sheaf complex. We prove this assertion in the next two lemmas.

Lemma 3.1. The map i_p induces an injective map of the first cohomology sheaves

$$i_p^* : \mathcal{H}^1(\mathcal{C}^{\cdot}_{Y_p/\Lambda_p}) \longrightarrow \mathcal{H}^1(\mathcal{C}^{\cdot}_{Y_p/\Lambda_p}). \tag{3.8}$$

□

Proof. First, we remark that there is an exact sequence of complexes

$$0 \longrightarrow \mathcal{C}^{\cdot}_{Y_p/\Lambda_p} \xrightarrow{i_p} \mathcal{C}^{\cdot}_{Y_p/\Lambda_p} \xrightarrow{r_p} \mathcal{C}^{\cdot}_{Y_{p+1}/\Lambda_{p+1}} \longrightarrow 0. \tag{3.9}$$

This exact sequence induces a long exact sequence in cohomology

$$\begin{aligned} \cdots \longrightarrow \mathcal{H}^k(\mathcal{C}^{\cdot}_{Y_p/\Lambda_p}) \xrightarrow{i_p^*} \mathcal{H}^k(\mathcal{C}^{\cdot}_{Y_p/\Lambda_p}) \xrightarrow{r_p^*} \mathcal{H}^k(\mathcal{C}^{\cdot}_{Y_{p+1}/\Lambda_{p+1}}) \\ \longrightarrow \mathcal{H}^{k+1}(\mathcal{C}^{\cdot}_{Y_p/\Lambda_p}) \longrightarrow \cdots . \end{aligned} \tag{3.10}$$

We assert that for any p, the sheaf $\mathcal{H}^0(\mathcal{C}^{\cdot}_{Y_p/\Lambda_p})$ can be identified with the sheaf $(\varphi^p)^{-1}(\mathcal{O}_{\Lambda_p})$. This implies, in particular, that the map

$$\mathcal{H}^0(\mathcal{C}^{\cdot}_{Y_p/\Lambda_p}) \xrightarrow{r_p^*} \mathcal{H}^0(\mathcal{C}^{\cdot}_{Y_{p+1}/\Lambda_{p+1}}) \tag{3.11}$$

is surjective.

Let U be a small open subset in Λ_p. Let $h \in \mathcal{O}_{Y_p}$ be a representative of a cohomology class in $\mathcal{H}^0(\mathcal{C}^{\cdot}_{Y_p/\Lambda_p})(U)$. Denote by X_i^p the restriction of the Hamilton vector field of F_i to Y_p. Since h is a coboundary, h commutes with the restriction of the F_i's to Λ_p, that is,

$$\{h, F_i\}_{|\Lambda_p} = L_{X_i^p} h = 0, \quad i = 1, \dots, n. \tag{3.12}$$

As f is nondegenerate and φ is a standard representative of a deformation of f, it follows that for a fixed value of λ, the X_i^p's generate the tangent space to $(\varphi^p)^{-1}(\lambda)$ at any point. Thus, equality (3.12) implies that h is constant along the fibres of φ^p. This proves the assertion.

Consequently, the long exact sequence splits at $\mathcal{H}^0(\mathcal{C}^{\cdot}_{Y_{p+1}/\Lambda_{p+1}})$ and we have the exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathcal{H}^1(\mathcal{C}^{\cdot}_{Y_p/\Lambda_p}) \xrightarrow{i_p^*} \mathcal{H}^1(\mathcal{C}^{\cdot}_{Y_p/\Lambda_p}) \xrightarrow{r_p^*} \mathcal{H}^1(\mathcal{C}^{\cdot}_{Y_{p+1}/\Lambda_{p+1}}) \\ &\longrightarrow \mathcal{H}^2(\mathcal{C}^{\cdot}_{Y_p/\Lambda_p}) \longrightarrow \cdots . \end{aligned} \tag{3.13}$$

This proves the lemma. ■

Lemma 3.2. For any $p \in \{0, \dots, k\}$, the restriction map r_p induces a surjective map of the first cohomology sheaves

$$r_p^* : \mathcal{H}^1(\mathcal{C}^{\cdot}_{Y_p/\Lambda_p}) \longrightarrow \mathcal{H}^1(\mathcal{C}^{\cdot}_{Y_{p+1}/\Lambda_{p+1}}). \tag{3.14}$$

□

Proof. Fix $p \in \{0, \dots, k\}$ and denote by ψ_i^p the restriction of $\partial_{\lambda_i} F$ to $\Lambda_p = \{\lambda_1 = \cdots = \lambda_p = 0\}$. It is readily verified that the coboundary of $\psi_i^p \in \mathcal{C}^1_{Y_p/\Lambda_p}$ vanishes for any $i \in \{1, \dots, k\}$.

We assert that there exists a small neighborhood U of the origin in Λ such that the cohomology classes of the ψ_i^p generate $\mathcal{H}^1(\mathcal{C}^{\cdot}_{Y_p/\Lambda_p})(\varphi^{-1}(U)) = \mathbb{R}^1\varphi_*\mathcal{C}^{\cdot}_{Y_p/\Lambda_p}(U)$.

Indeed, Theorem 2.5 says that $\mathbb{R}^1\varphi_*\mathcal{C}^{\cdot}_{Y_p/\Lambda_p}$ is a coherent sheaf on Λ_p and that its zero-fibre is isomorphic to $H^1(\mathcal{C}^{\cdot}_{Y_p/\Lambda_p,0})$. Thus $H^1(\mathcal{C}^{\cdot}_{Y_p/\Lambda_p,0})$ is an $\mathcal{O}_{\Lambda,0}$ module of finite type. Since Φ is infinitesimally $\mathcal{L}$-versal, the cohomology classes of the restrictions to $\lambda = 0$ of the germs at the origin of the ψ_i^p generate $H^1(C^{\cdot}_f)$. Thus, the Nakayama lemma implies that the cohomology classes of the ψ_i^p generate $H^1(\mathcal{C}^{\cdot}_{Y_p/\Lambda_p,0}) = (\mathbb{R}^1\varphi_*\mathcal{C}^{\cdot}_{Y_p/\Lambda_p})_0$. Therefore, by the coherence property, there exists a small neighborhood U of the origin in Λ such that cohomology classes of the ψ_i^p generate

$$\mathcal{H}^1(\mathcal{C}^{\cdot}_{Y_p/\Lambda_p})(\varphi^{-1}(U)) = (\mathbb{R}^1\varphi_*\mathcal{C}^{\cdot}_{Y_p/\Lambda_p})(U). \tag{3.15}$$

This proves the assertion.

Now, for $[m] \in \mathcal{H}^1(\mathcal{C}^{\cdot}_{Y_{p+1}/\Lambda_{p+1}})$, we write

$$[m] = \sum_{i=1}^{k} a_i [\psi_i^{p+1}]. \tag{3.16}$$

Then, $[m]$ is the image under r_p^* of $\sum_{i=1}^k a_i[\psi_i^p]$. This proves the lemma. ■

The exact sequences

$$0 \longrightarrow \mathcal{H}^1(\mathcal{C}^{\cdot}_{Y_p/\Lambda_p}) \longrightarrow \mathcal{H}^1(\mathcal{C}^{\cdot}_{Y_p/\Lambda_p}) \longrightarrow \mathcal{H}^1(\mathcal{C}^{\cdot}_{Y_{p+1}/\Lambda_{p+1}}) \longrightarrow 0 \tag{3.17}$$

imply that $(\lambda_1, \dots, \lambda_k)$ is a regular sequence of the $\mathcal{O}_{\lambda,0}$-module $(\mathbb{R}^1\varphi_*\mathcal{C}^{\cdot}_{Y/\Lambda})_0$. This means that $(\mathbb{R}^1\varphi_*\mathcal{C}^{\cdot}_{Y/\Lambda})_0$ is a Cohen-Macaulay $\mathcal{O}_{\Lambda,0}$-module, hence free (see, e.g., [3]). The coherence of the sheaf $(\mathbb{R}^1\varphi_*\mathcal{C}^{\cdot}_{Y/\Lambda})_0$ implies that in some neighborhood U of the origin, there is an isomorphism

$$\left(\mathbb{R}^1\varphi_*\mathcal{C}^{\cdot}_{Y/\Lambda}\right)_{|U} \approx \left(\mathcal{O}^{\mu}_{\Lambda}\right)_{|U} \tag{3.18}$$

for some nonnegative μ. This proves the first part of Theorem 2.7.

3.2 The dimension of $H^1(C^{\cdot}_f)$ is equal to β_1

Let $\bar{m}_1, \dots, \bar{m}_k$ be a basis of $H^1(C^{\cdot}_f)$.

Denote by $\Omega^{\cdot}_{Y/\Lambda}$ the relative de Rham complex

$$\Omega^k_{Y/\Lambda} = \left(\Omega^k_Y / \left(\varphi^*\Omega^1_\Lambda \wedge \Omega^{k-1}_Y\right)\right). \tag{3.19}$$

Let $X_1, \dots, X_n$ be the restriction of the Hamilton vector fields of the F_i's to Y.

The map

$$\begin{aligned} &\Omega^1_{Y/\Lambda} \longrightarrow \mathcal{C}^1_{Y/\Lambda}, \\ &\alpha \longmapsto \left(i_{X_1}\alpha, \dots, i_{X_n}\alpha\right) \end{aligned} \tag{3.20}$$

induces a map from the relative de Rham complex $\Omega^{\cdot}_{Y/\Lambda}$ to the complex of Lagrangian deformations $\mathcal{C}^{\cdot}_{Y/\Lambda}$. Here $i_{X_i}\alpha$ denotes the interior product of the one-form α with the vector field X_i.

As one easily sees, this map is an isomorphism at the smooth points of the fibres of $\varphi : Y \to \Lambda$ (see [10]).

We now take λ_0 in the small neighborhood $U \subset \Lambda$, such that the fibre of φ at λ_0 is smooth. The following lemma concludes the proof of the theorem.

Lemma 3.3. The rank of the free $\mathcal{O}_{\Lambda,\lambda_0}$-module $(\mathbb{R}^1\varphi_*\mathcal{C}^{\cdot}_{Y/\Lambda})_{\lambda_0}$ equals the dimension of $H^1(L_{\lambda_0}, \mathbb{C})$. □

Proof. Since the fibre of φ at λ_0 is smooth, we have an isomorphism

$$\left(\mathbb{R}^1\varphi_*\mathcal{C}^{\cdot}_{Y/\Lambda}\right)_{\lambda_0} \approx \left(\mathbb{R}^1\varphi_*\Omega^{\cdot}_{Y/\Lambda}\right)_{\lambda_0}. \tag{3.21}$$

Then, the freeness of the $\mathcal{O}_{\Lambda,\lambda_0}$-module

$$\left(\mathbb{R}^1\varphi_*\mathcal{C}^{\cdot}_{Y/\Lambda}\right)_{\lambda_0} \approx \mathcal{O}^{\mu}_{\Lambda,\lambda_0} \tag{3.22}$$

implies the isomorphism of $\mathbb{C}$-vector spaces

$$\left(\left(\mathbb{R}^1\varphi_*\Omega^{\cdot}_{Y/\Lambda}\right)_{\lambda_0}/\mathcal{M}_{\Lambda,\lambda_0}\left(\mathbb{R}^1\varphi_*\Omega^{\cdot}_{Y/\Lambda}\right)_{\lambda_0}\right) \approx \mathbb{H}^1\left(L_{\lambda_0},\Omega^{\cdot}_{L_{\lambda_0}}\right). \tag{3.23}$$

Here, $\mathcal{M}_{\Lambda,\lambda_0}$ is the maximal ideal of $\mathcal{O}_{\Lambda,\lambda_0}$ and $\Omega^{\cdot}_{\lambda_0}$ is the de Rham complex on $\varphi^{-1}(\lambda_0)$. Since the de Rham complex $\Omega^{\cdot}_{\lambda_0}$ is a resolution of the constant sheaf $\mathbb{C}$, we get that

$$\mathbb{H}^1\left(L_{\lambda_0},\Omega^{\cdot}_{L_{\lambda_0}}\right) \approx H^1\left(L_{\lambda_0},\mathbb{C}\right). \tag{3.24}$$

Thus the rank of $(\mathbb{R}^1\varphi_*\mathcal{C}^{\cdot}_{Y/\Lambda})_{\lambda_0}$ is equal to the dimension of $H^1(L_{\lambda_0},\mathbb{C})$.

This concludes the proof of the lemma and the proof of Theorem 2.7. ∎

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